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# SU(3) Breaking in K and K<sup>\*</sup> Distribution Amplitudes

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## Abstract:

We calculate the decay couplings and first two Gegenbauer moments of the leading twist light-cone distribution amplitudes of K and K<sup>\*</sup> from QCD sum rules, including NLO perturbative effects.

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# 1 Introduction and Motivation

Hadronic light-cone distribution amplitudes (DAs) of leading twist play an essential rôle in the QCD description of hard exclusive processes. They can be obtained from the hadron’s Bethe-Salpeter equation by integration over the transverse momentum distribution, but keeping track of the longitudinal momentum fraction  $u$ :

$$\phi(u) \sim \int_{k_\perp^2 < \mu^2} d^2 k_\perp \phi(u, k_\perp).$$

DAs enter the amplitudes of processes to which collinear factorisation theorems apply and which notably include “classical” applications like the EM form factor of the pion or the  $\gamma\gamma^*\pi$  transition form factor, which were first discussed in the seminal papers by Brodsky, Lepage and others [1]. More recently, collinear factorisation has been shown to apply, to leading order in an expansion in  $1/m_b$ , also to a large class of nonleptonic B decays [2], which has opened a new and exciting area of applications of meson DAs: nonleptonic B decays, and in particular CP asymmetries in these decays, are currently being studied at the B factories BaBar and Belle and are expected to yield essential information about the pattern of CP violation and potential sources of flavour violation beyond the SM. DAs also enter crucially SCET, the soft-collinear effective theory [3], which aims to provide a unified theoretical framework for the factorisation of both hard-collinear effects, relevant for all hard exclusive QCD processes, and soft effects, which occur in processes involving heavy mesons.

For light mesons, on which we shall concentrate in this paper, leading twist DAs can be interpreted as probability amplitudes of finding the meson in a state with a minimum number of Fock constituents and at small transverse separation which provides an ultra-violet cut-off. The dependence on this cut-off  $\mu$  is given by Brodsky-Lepage evolution equations and can be calculated in perturbative QCD, while the DAs at a certain low scale provide the necessary nonperturbative input for a rigorous QCD treatment of exclusive reactions with large momentum transfer.

As exclusive QCD processes come with much smaller cross-sections or branching ratios than inclusive processes, they are not as well studied experimentally as their inclusive counterparts. The overall normalisations of DAs are given by local hadronic matrix elements, essentially decay constants, which are partly accessible experimentally, partly have to be calculated from theory. Information on the shape of DAs comes mostly from theory and has been the subject of numerous studies within various nonperturbative approaches. There are a few, mostly exploratory, studies of the second momentum of the pion DA in lattice QCD, Ref. [4], but the results are not yet refined enough to be relevant for phenomenology. Most of the existing information on light meson DAs comes from QCD sum rules, whose application to the study of moments of DAs has been pioneered by Chernyak and Zhitnitsky. Their review on the “Asymptotic Behavior Of Exclusive Processes In QCD”, Ref. [5], is still a very useful source of information, despite having been written 20 years ago. Later analyses switched from the calculation of moments to that of *Gegenbauer* moments which, as we shall discuss in Sec. 2, are more appropriate for the study of DAs. The leading-twist DA of the pion was analysed in Ref. [6], while those of the  $\rho$  were studied in Ref. [7]. The first paper to study SU(3) breaking effects in

moments was Ref. [8], while the first two Gegenbauer moments of the  $K^*$  and  $\Phi$  meson DAs were calculated in Ref. [9].

The aim of the present paper is to provide a careful analysis of SU(3) breaking effects in leading-twist  $K$  and  $K^*$  DAs, using QCD sum rules. The motivation for this study is the need for a more accurate assessment of these effects than presented in the papers quoted above. This need is driven by the fact that DAs are crucial ingredients in the analysis of B decays of type  $B \rightarrow \pi K$  etc., which are presently being measured at the B factories. In the future, accurate information on SU(3) breaking effects in DAs will also be required for assessing the reliability of phenomenological methods for relating, via U-spin symmetry, amplitudes of  $B_d$  decays to those of  $B_s$  decays which will be measured at the Tevatron and the LHC. SU(3) breaking enters these processes in essentially three different ways:

- via an asymmetry of the DAs under the exchange of quark and antiquark;
- via SU(3) breaking in the symmetric parts of the DA, in particular the overall normalisation;
- via form factors, e.g.  $F^{B \rightarrow \pi}$  vs.  $F^{B \rightarrow K}$ .

Although in the present paper we will concentrate on the first two manifestations of SU(3) breaking, our results can be used immediately for an update of  $B \rightarrow K, K^*$  form factors using QCD sum rules on the light-cone (cf. [10] for the present state of the art).

Our paper is organised as follows: Sec. 2 contains the definition of all relevant DAs as well as a discussion of their behaviour under a change of renormalisation scale. In Sec. 3 we discuss QCD sum rules for the DAs, obtain numerical results and compare with the results of other studies. Section 4 presents a summary and our conclusions. Issues of a more technical nature are discussed in the appendices.

## 2 General Framework

### 2.1 Definitions

We define the light-cone DAs via matrix elements of quark-antiquark gauge-invariant nonlocal operators at light-like separations  $z_\mu$  with  $z^2 = 0$  [5]. For definiteness we consider the  $K^{(*)-}$  meson distributions; the DAs of the neutral mesons containing an s quark just involve a trivial isospin factor  $1/\sqrt{2}$  in the overall normalisation. For mesons containing an s antiquark, one has  $\phi_{(\bar{s}q)}(u) = \phi_{(\bar{q}s)}(1-u)$ . The complete set of distributions to leading-twist accuracy involves three DAs (we use the notation  $\hat{z} = z^\mu \gamma_\mu$  for arbitrary 4-vectors  $z$ ):

$$\langle 0 | \bar{u}(z) \hat{z} \gamma_5 [z, 0] s(0) | K^-(q) \rangle = i f_K (q \cdot z) \int_0^1 du e^{-i\bar{u}(q \cdot z)} \phi_K(u),$$

$$\langle 0 | \bar{u}(z) \hat{z} [z, 0] s(0) | K^{*-}(q, \lambda) \rangle = (e^{(\lambda)} z) f_K^\parallel m_{K^*} \int_0^1 du e^{-i\bar{u}(q \cdot z)} \phi_K^\parallel(u),$$

$$\langle 0 | \bar{u}(z) \sigma_{\mu\nu} [z, 0] s(0) | K^{*-}(q, \lambda) \rangle = i(e_{\mu}^{(\lambda)} q_{\nu} - e_{\nu}^{(\lambda)} q_{\mu}) f_K^{\perp}(\mu) \int_0^1 du e^{-i\bar{u}(q \cdot z)} \phi_K^{\perp}(u), \quad (2.1)$$

with the Wilson-line

$$[z, 0] = \text{Pexp} \left[ ig \int_0^1 d\alpha z^{\mu} A_{\mu}(\alpha z) \right]$$

inserted between quark fields to render the matrix elements gauge-invariant. In the above definitions,  $e_{\nu}^{(\lambda)}$  is the polarization vector of a vector meson with polarisation  $\lambda$ ; there are two DAs for vector mesons, corresponding to longitudinal and transverse polarisation, respectively. The integration variable  $u$  is the meson momentum fraction carried by the quark,  $\bar{u} \equiv 1 - u$  the momentum fraction carried by the antiquark. The normalisation constants  $f_K$  are defined by the local limit of Eqs. (2.1) and chosen in such a way that

$$\int_0^1 du \phi(u) = 1 \quad (2.2)$$

for all the three distributions  $\phi_K, \phi_K^{\parallel}, \phi_K^{\perp}$ .

## 2.2 Conformal Expansion and Renormalisation

The conformal expansion of light-cone distribution amplitudes is analogous to the partial wave expansion of wave functions in standard quantum mechanics. In conformal expansion, the invariance of massless QCD under conformal transformations is the equivalent of rotational symmetry in quantum mechanics, where, for spherically symmetric potentials, the partial wave decomposition serves to separate angular degrees of freedom from radial ones. All dependence on the angular coordinates is included in spherical harmonics which form an irreducible representation of the group  $O(3)$ , and the dependence on the single remaining radial coordinate is governed by a one-dimensional Schrödinger equation. Similarly, the conformal expansion of distribution amplitudes in QCD aims to separate longitudinal degrees of freedom from transverse ones. All dependence on the longitudinal momentum fractions is described by orthogonal polynomials that form an irreducible representation of the so-called collinear subgroup of the conformal group,  $SL(2, \mathbb{R})$ , describing Möbius transformations on the light-cone. The transverse-momentum dependence (scale-dependence) is governed by simple renormalisation group equations: the different partial waves, labelled by different “conformal spins”, behave independently and do not mix with each other. Since the conformal invariance of QCD is broken by quantum corrections, mixing between different terms of the conformal expansion is absent only to leading logarithmic accuracy. Still, conformal spin is a good quantum number in hard processes, up to small corrections of order  $\alpha_s^2$ . The application of conformal symmetry to the study of exclusive processes to leading twist has become a vast field whose current status is reviewed in Ref. [11].

Since quark mass terms break the conformal symmetry of the QCD Lagrangian explicitly one might, at first glance, expect difficulties to incorporate  $SU(3)$  breaking corrections into the formalism. In fact, however, the inclusion of quark mass corrections is straightforward and generates two types of effects. First, matrix elements of conformal operators are modified and in general do not have the symmetry of the massless theory. This is

not a “problem”, since the conformal expansion is designed to simplify the transverse momentum dependence of the wave functions by relating it to the scale dependence of the relevant operators. This dependence is given by operator anomalous dimensions which are not affected by quark masses, provided they are smaller than the scales involved. Second, new higher twist operators arise, in which quark masses multiply operators of lower twist. These additional operators generate contributions to higher twist DAs, which have been discussed in [9] and are irrelevant for the present investigation.

The conformal expansion of DAs is especially simple when each constituent field has a fixed (Lorentz) spin projection onto the light-cone. In this case, its conformal spin is

$$j = \frac{1}{2}(l + s), \quad (2.3)$$

where  $l$  is the canonical dimension and  $s$  the (Lorentz) spin projection. Multi-particle states built of constituent fields can be expanded in increasing conformal spin: the lowest possible spin equals the sum of spins of the constituents, and its “wave function” is given by the product of one-particle states. This state is nondegenerate and cannot mix with other states because of conformal symmetry. Its evolution is given by a simple renormalisation group equation and the corresponding anomalous dimension is the smallest one in the spectrum. This state is the only one to survive in the limit  $Q^2 \rightarrow \infty$  and is usually referred to as “asymptotic distribution amplitude”.

As for the leading-twist quark-antiquark DAs we are interested in, the partial wave expansion reads

$$\phi(u) = 6u\bar{u} \sum_{n=0}^{\infty} a_n C_n^{3/2}(2u-1), \quad (2.4)$$

where  $C_n^{3/2}(2u-1)$  are Gegenbauer polynomials. The dimension of quark fields is  $l = 3/2$  and the leading twist distribution corresponds to positive spin projection  $s = +1/2$  for both the quark and the antiquark. Thus, according to (2.3), the conformal spin of each field is  $j_q = j_{\bar{q}} = 1$ ; the asymptotic distribution amplitude is  $6u\bar{u}$ . The Gegenbauer polynomials correspond to contributions with higher conformal spin  $j + n$  and are orthogonal over the weight function  $6u\bar{u}$ .

Note that  $a_0 = 1$  due to the normalisation condition (2.2). In the limit of exactly massless quarks only terms with even  $n$  survive in Eq. (2.4) because of G-parity invariance. The conformal expansion, however, can be performed at the operator level and is disconnected from particular symmetries of states such as G-parity. The expansion (2.4) is, therefore, valid for arbitrary  $n$ , and it is precisely the odd contributions to the expansion which, being proportional to manifestly SU(3) breaking effects like the difference of quark masses, induce the most tangible SU(3) breaking effects.

As mentioned before, conformal invariance implies that partial waves with different conformal spin do not mix under renormalisation to leading-order accuracy, which means that the Gegenbauer moments  $a_n$  in (2.4) renormalise multiplicatively:

$$a_n(\mu) = a_n(\mu_0) \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{(\gamma_{(n)} - \gamma_{(0)})/\beta_0} \quad (2.5)$$

with  $\beta_0 = 11 - (2/3)n_f$ . The one-loop anomalous dimensions are [12]

$$\begin{aligned}\gamma_{(n)} = \gamma_{(n)}^{\parallel} &= C_F \left( 1 - \frac{2}{(n+1)(n+2)} + 4 \sum_{j=2}^{n+1} 1/j \right), \\ \gamma_{(n)}^{\perp} &= C_F \left( 1 + 4 \sum_{j=2}^{n+1} 1/j \right).\end{aligned}\tag{2.6}$$

$\gamma_{(0)}$  is the anomalous dimension of the local current and vanishes for vector and axialvector currents. Note that the DAs of the pseudoscalar  $K$  and the longitudinally polarised  $K^*$  have the same anomalous dimensions. As the  $\gamma$ 's are positive and increase with  $n$ , the effect of running to large scales is to damp the effects of higher-order Gegenbauers, so that the DAs approach their asymptotic shape. From the above-stated scaling-properties, it is evident that a discussion of the shape of DAs is most conveniently done in terms of Gegenbauer moments.

## 3 QCD Sum Rules for Moments

### 3.1 QCD Sum Rules

As QCD sum rules constitute an established method for the calculation of static hadronic matrix elements like decay constants (as opposed to dynamical quantities like form factors), we refrain from delving into a comprehensive explanation of technicalities, for which we refer to the original papers [13] and to recent reviews [14] instead. The key feature of the method is the use of analyticity to relate the local short-distance operator product expansion (OPE) of a correlation function of two currents,

$$\Pi = i \int d^4y e^{iqy} \langle 0 | T J_1(y) J_2(0) | 0 \rangle = \sum_n C_n(q^2) \langle O_n \rangle \equiv \Pi^{\text{OPE}} \tag{3.1}$$

around  $y = 0$  (as opposed to a light-cone expansion around  $y^2 = 0$ , which is appropriate for form factor calculations, cf. [10]) valid for  $Q^2 \equiv -q^2 \ll 0$ , to its dispersion relation in terms of hadronic contributions,

$$\Pi = \int_0^\infty ds \frac{\rho(s)}{s - q^2 - i0} \equiv \Pi^{\text{had}}, \tag{3.2}$$

where  $\rho(s)$  is the spectral density of the correlation function along its physical cut. The OPE yields a series of local operators of increasing dimension whose expectation values  $\langle O_n \rangle$  in the nonperturbative (physical) vacuum are the so-called condensates. In the sum rules analysed in this paper, we take into account the condensates listed in Tab. 1. The representation of the correlation function in terms of hadronic matrix elements can be written as

$$\rho(s) = f \delta(s - m_M^2) + \rho^{\text{cont}}(s),$$

$\langle \bar{q}q \rangle = (-0.24 \pm 0.01)^3 \text{ GeV}^3$	$\langle \bar{s}s \rangle = (0.8 \pm 0.1) \langle \bar{q}q \rangle$
$\langle \frac{\alpha_s}{\pi} G^2 \rangle = (0.012 \pm 0.006) \text{ GeV}^4$	
$\langle \bar{q}\sigma g G q \rangle = (0.8 \pm 0.1) \text{ GeV}^2 \langle \bar{q}q \rangle$	$\langle \bar{s}\sigma g G s \rangle = (0.8 \pm 0.1) \langle \bar{q}\sigma g G q \rangle$
$\bar{m}_s(2 \text{ GeV}) = (110 \pm 20) \text{ MeV} \longleftrightarrow \bar{m}_s(1 \text{ GeV}) = (149 \pm 27) \text{ MeV}$	
$\alpha_s(1 \text{ GeV}) = 0.513 \longleftrightarrow \Lambda_{\text{QCD}}^{(3)\text{NLO}} = 372 \text{ MeV}$	

Table 1: Input parameters for sum rules at the renormalisation scale  $\mu = 1 \text{ GeV}$ . The value of  $m_s$  is obtained from quenched lattice calculations as summarised in [15].

where  $m_M$  is the mass of the lowest-lying state coupling to the currents  $J_{1,2}$  and  $\rho^{\text{cont}}$  parametrises all contributions to the correlation function apart from the ground state.  $f$ , the residue of the ground state pole is the quantity one wants to determine. A QCD sum rule that allows one to do so is obtained by equating the representations (3.1) and (3.2) and implementing the following (model) assumptions:

- $\rho^{\text{cont}}$  is approximated by the spectral density obtained from the OPE above a certain threshold, i.e.  $\rho^{\text{cont}} \rightarrow \rho^{\text{OPE}}(s)\theta(s - s_0)$  with  $s_0 \approx (m_M + \Delta)^2$  being the continuum threshold, where  $\Delta \sim O(\Lambda_{\text{QCD}})$  is an excitation energy to be determined within the method. This assumption relies on the validity of global quark-hadron duality;
- instead of the weight-functions  $1/(q^2)^n$  and  $1/(s - q^2)$ , one uses different weight-functions which are optimised to (exponentially) suppress effects of  $\rho(s)$  for large values of  $s$  and at the same time also suppress high-dimensional condensates by factorials. This is achieved by Borel transforming the correlation function:  $\mathcal{B} 1/(s - q^2) = 1/M^2 \exp(-s/M^2)$ . A window of viable values of the Borel parameter  $M^2$  and the continuum threshold  $s_0$  has to be determined within the method itself by looking for a maximum region of minimum sensitivity (a plateau) in both  $M^2$  and  $s_0$ ;
- the OPE of  $\Pi$  can be truncated after a few terms. As we shall see (and as is well known), this condition is fulfilled only for low Gegenbauer moments.

After subtraction of the integral over  $\rho^{\text{OPE}}$  above  $s_0$  from both sides, the final sum rule reads

$$\mathcal{B}_{\text{sub}} \Pi^{\text{OPE}} \equiv \frac{1}{M^2} \int_0^{s_0} ds e^{-s/M^2} \rho^{\text{OPE}}(s) = \frac{f}{M^2} e^{-m_M^2/M^2}, \quad (3.3)$$

which gives the hadronic quantity  $f$  as a function of the Borel parameter  $M^2$  and the continuum threshold  $s_0$  (and the condensates and short-distance parameters from the OPE).

Even Gegenbauer moments can be determined from diagonal correlation functions of type

$$i \int d^4 y e^{iqy} \langle 0 | T \bar{q}(y) \Gamma s(y) \bar{s}(0) \Gamma [0, z] q(z) | 0 \rangle, \quad (3.4)$$



with a suitably chosen Dirac structure  $\Gamma$ . This form of the correlation function differs from Refs. [5, 7], where local operators with an arbitrary number of covariant derivatives were used rather than the nonlocal operator  $\bar{s}(0)\Gamma[0, z]q(z)$ . We find the calculation with nonlocal operators very convenient, as it allows one to calculate all moments in one go. Specifying for instance to  $K_{\parallel}^*$ , the sum rule reads

$$\mathcal{B}_{\text{sub}} \Pi^{\text{OPE}} = (f_K^{\parallel})^2 e^{-m_{K^*}^2/M^2} \frac{1}{M^2} \int_0^1 du e^{i\bar{u}(q \cdot z)} \phi_K^{\parallel}(u), \quad (3.5)$$

where also  $\Pi^{\text{OPE}}$  is expressed as integral over  $u$ , which naturally emerges as Feynman parameter in the calculation, and comes with the same weight function  $\exp(i\bar{u}(q \cdot z))$ . Sum rules for individual Gegenbauer moments are obtained by expanding both sides in powers of  $(q \cdot z)$ , or effectively replacing

$$e^{i\bar{u}(q \cdot z)} \rightarrow C_n^{3/2}(2u - 1), \quad \int_0^1 du e^{i\bar{u}(q \cdot z)} \phi_K^{\parallel}(u) \rightarrow \frac{3(n+1)(n+2)}{2(2n+3)} a_n^{\parallel}.$$

For odd moments, one analyses nondiagonal correlation functions of type

$$i \int d^4y e^{iqy} \langle 0 | T \bar{q}(y) \Gamma_2 s(y) \bar{s}(0) \Gamma_1 [0, z] q(z) | 0 \rangle \quad (3.6)$$

with structures  $\Gamma_1$  and  $\Gamma_2$  of opposite chirality [8, 5]. The  $\Gamma_i$  appropriate for  $K$ ,  $K_{\parallel}^*$  and  $K_{\perp}^*$  are given in App. A, where we also give complete expressions for perturbative and quark-condensate contributions to  $O(\alpha_s)$ , as well as tree-expressions for the dimension 5 mixed condensate.

Formulas for the correlation functions and Gegenbauer moments are collected in the appendices. Note in particular the scaling of different contributions to the Gegenbauer moments in  $n$ , the order of the moment: nonperturbative terms increase with positive powers of  $n$  with respect to the perturbative contribution. For large  $n$ , this behaviour upsets the usual hierarchy of contributions to the OPE, where nonperturbative terms are expected to be a moderately sized correction to the leading term. The origin of this behaviour can be easily understood from the fact that in the local expansion the vacuum fields have exactly zero momentum, which yields the  $\delta$ -function terms in (A.3) and (A.4). This amounts to a multipole expansion of the DA around its endpoints, which is justified if one is only interested in gross features of the DA, like the first few moments, but which is clearly inappropriate for extracting more detailed information on the shape. In the present paper we adopt the viewpoint that at least the first two Gegenbauer moments can be reliably obtained from local sum rules of the type discussed above.

### 3.2 A Short Discussion of Decay Constants and Lattice Results

The decay constant of the pseudoscalar K meson is well known from  $K^+ \rightarrow \mu^+ \nu_{\mu} (+\gamma)$  and quoted as [16]

$$f_K = (159.8 \pm 1.3) \text{ MeV}.$$

The decay constant  $f_K^\parallel$  can be extracted from the branching ratio of  $\tau^- \rightarrow K^{*-} \nu_\tau$  via

$$B(\tau^- \rightarrow K^{*-} \nu_\tau) = \frac{G_F^2 m_\tau |V_{us}|^2}{8\pi} \tau_\tau m_{K^*}^2 (f_K^\parallel)^2 \left(1 + \frac{m_\tau^2}{2m_{K^*}^2}\right) \left(1 - \frac{m_{K^*}^2}{m_\tau^2}\right)^2.$$

With  $|V_{us}| = 0.220$  and the other parameters taken from [16], one finds

$$f_K^\parallel = (217 \pm 5) \text{ MeV}.$$

It is interesting to compare this value with a “postdiction” from QCD sum rules. We use the sum rule quoted as (C.4) in [9] with the input parameters as in Tab. 1. From the result plotted in Fig. 1 we conclude

$$f_K^{\parallel (\text{SR})} = (225 \pm 7) \text{ MeV}$$

and  $s_0 = (1.8 \pm 0.1) \text{ GeV}^2$ . The plateau in  $M^2$  extends from 1 to 2  $\text{GeV}^2$ . Although the error only includes the impact of varying the input parameters of Tab. 1 within their respective ranges and does not account for any systematic uncertainties, the agreement with the experimental result is remarkably good.

The value of the third constant,  $f_K^\perp$ , has not been measured yet, but has been calculated in quenched lattice QCD and from QCD sum rules. The lattice result for the ratio of the  $K^*$  decay constants is [17]

$$\frac{f_K^\perp(2 \text{ GeV})}{f_K^\parallel} = 0.739(17).$$

From QCD sum rules, on the other hand, we find, cf. Fig. 2,

$$\frac{f_K^\perp(1 \text{ GeV})}{f_K^\parallel} = 0.84(03) \quad \longleftrightarrow \quad \frac{f_K^\perp(2 \text{ GeV})}{f_K^\parallel} = 0.74(03),$$

using the formulas (C.4) and (C.5) in [9], which include NLO radiative corrections. The optimum continuum threshold for  $f_K^\perp$  turns out to be  $s_0 = (1.2 \pm 0.1) \text{ GeV}^2$ . The fact that  $s_0$  for  $K_\perp^*$  is smaller than for  $K_\parallel^*$  is in agreement with the findings of Ref. [7] and related to the fact that, with the Dirac structure  $\Gamma$  chosen as specified in App. B, the sum rule does not only receive contributions from  $K^*$ , but also from the ground state in the opposite parity channel, the  $1^+$  state  $K_1(1270)$ . This contribution has to be included in the continuum, which results in a low value of  $s_0$ . Note that this is not an external condition imposed by us when evaluating the sum rule, but emerges naturally from the criterion of stability and the presence of a plateau in  $M^2$ . We consider the ratio of decay constants rather than the constants themselves, as effects of unknown higher order corrections are expected to cancel in the ratio. The sum rules are evaluated at the low scale 1 GeV and the scaling from 1 to 2 GeV is done using NLO renormalisation group improvement. Note that the dependence on the Borel parameter largely cancels in the ratio of decay constants and also the dependence on  $s_0$  is rather mild. The agreement with the lattice results is remarkably good. Using the experimental value for  $f_K^\parallel$ , QCD sum rules hence predict, to NLO accuracy and including 2-loop running,

$$\begin{aligned} f_K^\perp(1 \text{ GeV}) &= (182 \pm 10) \text{ MeV}, & f_K^\perp(2 \text{ GeV}) &= (160 \pm 9) \text{ MeV}, \\ f_K^\perp(4.8 \text{ GeV}) &= (149 \pm 8) \text{ MeV}. \end{aligned} \tag{3.7}$$

This updates the result obtained in Ref. [9].

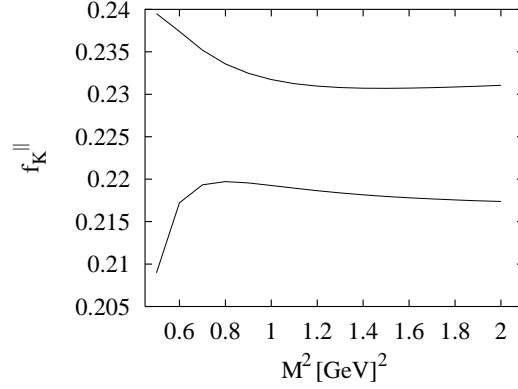


Figure 1:  $f_K^{\parallel}$  from QCD sum rules as function of the Borel parameter. The spread between the two curves corresponds to the uncertainty induced by varying the input parameters within their error margins.

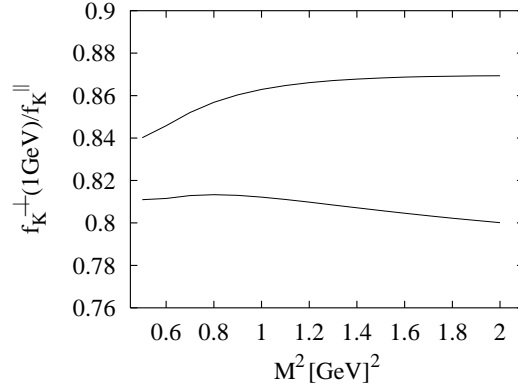


Figure 2: The ratio  $f_K^{\perp}(1 \text{ GeV})/f_K^{\parallel}$  as function of the Borel parameter  $M^2$ . Interpretation of curves as in previous figure.  $s_0$  is varied between 1.1 and 1.3  $\text{GeV}^2$  for  $f_K^{\perp}$  and 1.7 and 1.9  $\text{GeV}^2$  for  $f_K^{\parallel}$ .

### 3.3 The Gegenbauer Moment $a_1$

Let us now calculate the first moments of the  $K$  and  $K^*$  DAs. Previous determinations go back to Refs. [5, 9], where the following values were obtained:

$$a_1(1 \text{ GeV}) = 0.17, \quad a_1^{\parallel}(1 \text{ GeV}) = 0.19 \pm 0.05, \quad a_1^{\perp}(1 \text{ GeV}) = 0.20 \pm 0.05.$$

These results are valid at LO in QCD and were obtained using QCD sum rules which, unfortunately, partly suffered from mistakes. In particular, we find a different *sign* of the perturbative contribution w.r.t. the formulas given in Ref. [5, 9]. We have carefully checked that the sign we obtain is indeed the correct one: for the  $n$ th Gegenbauer moment, the leading order perturbative contribution is the same for all three correlation functions

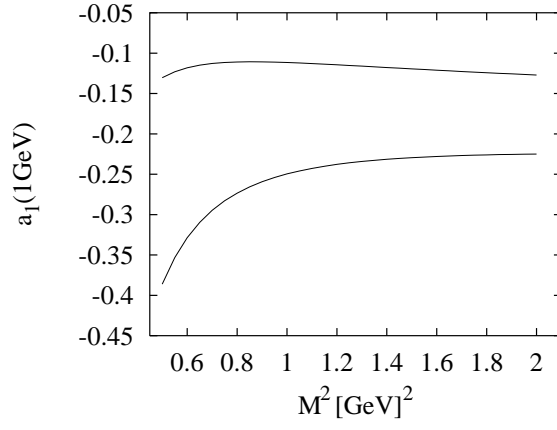


Figure 3:  $a_1(1\text{GeV})$  as function of the Borel parameter  $M^2$ . The spread between the two curves corresponds to the uncertainty induced by varying the input parameters within their error margins.  $s_0$  is fixed at  $1.8 \text{ GeV}^2$ .

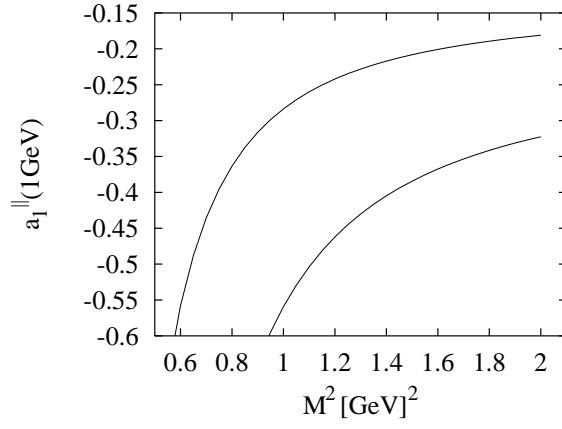


Figure 4:  $a_1^{\parallel}(1\text{GeV})$ ; notations and conventions as above.

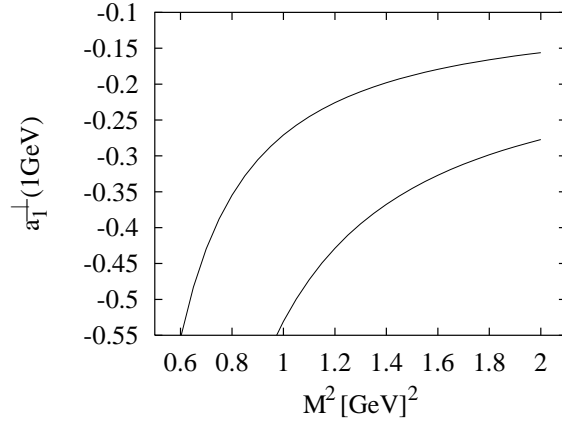


Figure 5:  $a_1^{\perp}(1\text{GeV})$ ;  $s_0$  is fixed at  $1.2 \text{ GeV}^2$ ; other notations and conventions as above.

and is given by

$$\frac{3}{4\pi^2} m_s \frac{1}{M^2} \left(1 - e^{-s_0/M^2}\right) \int_0^1 C_n^{3/2}(2u-1)\bar{u} = \frac{3}{4\pi^2} m_s \frac{1}{M^2} \left(1 - e^{-s_0/M^2}\right) \left(-\frac{1}{2}\right)^n.$$

It appears that the factor  $(-1)^n$  has been missed in [5]. Due to the change in sign of the perturbative contribution, the leading order sum rule becomes numerically unstable: perturbation theory and the mixed condensate yield negative contributions, the quark condensate a positive one, and the sum is close to zero. This is in stark contrast to the statement made in Ref. [5], according to which the sum rule is dominated by the quark condensate contribution and yields a positive result for all  $a_1$ .

The different signs of individual contributions, together with the fact that none of them is numerically dominant, entails that no meaningful result can be extracted from the leading order sum rule. The situation improves, however, if one includes radiative corrections. We have calculated  $O(\alpha_s)$  corrections to both the perturbative and the quark condensate contribution and find that they reduce the size of the quark condensate contribution, but increase that of the perturbative contribution. As a result, the perturbative contribution becomes numerically dominant, as is actually expected for a “good” sum rule. By varying all the input parameters of Tab. 1 within their respective ranges we obtain

$$a_1(1 \text{ GeV}) = -0.18 \pm 0.09, \quad a_1^{\parallel}(1 \text{ GeV}) = -0.4 \pm 0.2, \quad a_1^{\perp}(1 \text{ GeV}) = -0.34 \pm 0.18. \quad (3.8)$$

The Borel window is taken to be 1 to 2 GeV<sup>2</sup>, as motivated by the results of the previous subsection. Note that the l.h.s. of the sum rules for  $a_1^{\parallel,\perp}$  contains the factor  $f_K^{\parallel} f_K^{\perp}$ , which we substitute by their respective sum rules instead of using the values determined in the previous subsection, the reason being that one expects unknown higher order corrections to cancel in the ratio. The dependence of the individual sum rules on the input parameters is plotted in Figs. 3, 4 and 5. The error bars in (3.8) are rather large, which is due to the fact that the first Gegenbauer moments are explicitly proportional to SU(3) breaking quantities, i.e.  $m_s$  or the difference of condensates, e.g.  $\langle \bar{q}q \rangle - \langle \bar{s}s \rangle$ , which come with a considerable uncertainty.

We would also like to add that the use of nondiagonal sum rules has been met with criticism. It has been argued in Ref. [18] that these sum rules may suffer from large contributions of higher resonances or, in the case of the  $K$ , from instanton contributions to the pseudoscalar current. We can actually estimate the amount of possible contamination of these sum rules by studying their local limit, yielding  $a_0$ , which is 1 by definition. In Fig. 6 we plot the sum rules for  $a_0$  for both the  $K$  and the  $K^*$  (there is only one sum rule for the  $K^*$ ), for central values of the input parameters,  $s_0 = 1.2 \text{ GeV}^2$  for  $K^*$ ,  $s_0 = 1.8 \text{ GeV}^2$  for the  $K$ , and replacing  $f_K^{\perp,\parallel}$  in the denominator by their respective sum rules. The plot shows no sizeable contamination for the  $K$ , and a moderate one for the  $K^*$ ; both results lie in the ball-park of the expected accuracy of QCD sum rules and come with uncertainties from the input parameters, in particular for  $a_0$  of the  $K$ , which is directly proportional to  $m_s$ . Radiative corrections are important in both cases and drag the result closer to  $a_0 = 1$ . This result strengthens our confidence in the suitability of nondiagonal sum rules for extracting meaningful values of  $a_1$ .

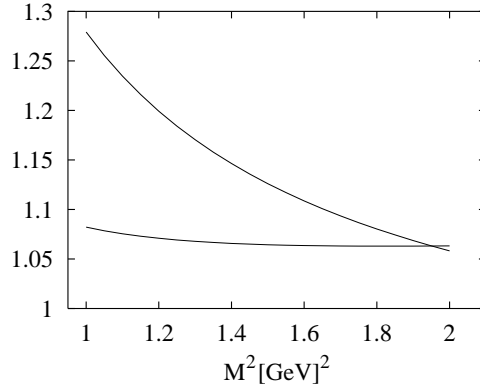


Figure 6:  $a_0$  from nondiagonal sum rules for  $K$  (lower curve) and  $K^*$  (upper curve) for central values of input parameters.

### 3.4 The Gegenbauer Moment $a_2$

Sum rules for even moments and the effects of SU(3) breaking have been studied in various papers, e.g. [8, 5, 9]. None of these papers, however, does contain a complete set of sum rules for Gegenbauer moments. Moreover, while checking our two-loop and nonperturbative calculations against the formulas collected in Ref. [9], we found that the terms in  $m_s \langle \bar{s} \sigma g G s \rangle$ , given correctly in Ref. [5] for the moments of  $K^{(*)}$ , have not been correctly translated into Gegenbauer moments in Ref. [9]; we have calculated these contributions anew for all three DAs, using nonlocal currents, and confirm the results of Ref. [5]. We have also recalculated the four-quark contributions and confirm the expressions quoted in [9]. The complete set of (hopefully) correct sum rules is given in App. B.

With the input parameters from Tab. 1 we obtain

$$a_2(1 \text{ GeV}) = 0.16 \pm 0.10, \quad a_2^{\parallel}(1 \text{ GeV}) = 0.09 \pm 0.05, \quad a_2^{\perp}(1 \text{ GeV}) = 0.13 \pm 0.08 \quad (3.9)$$

by varying all the input parameters of Tab. 1 within their respective ranges. These numbers have to be compared with those quoted in the first reference in [10] and in Ref. [9]:

$$a_2(1 \text{ GeV}) = 0.2, \quad a_2^{\parallel}(1 \text{ GeV}) = 0.06 \pm 0.06, \quad a_2^{\perp}(1 \text{ GeV}) = 0.04 \pm 0.04.$$

The discrepancy to (3.9) is mainly due to the wrong expressions for the contribution of the  $\langle \bar{s} \sigma g G s \rangle$  condensate used in [9].

Like for  $a_1$ , the relative errors of the moments (3.9) are considerably larger than the ones quoted for the decay constants. The reason is the absence of a stability plateau in  $M^2$  as the perturbative contribution is of  $O(\alpha_s)$  and small. For the extraction of the Gegenbauer moments we thus have to rely on the optimum values of  $M^2$ , 1 to 2 GeV<sup>2</sup>, and  $s_0$  determined from the sum rules for the decay constants.

### 3.5 Models for Distribution Amplitudes

Nearly all models for DAs rely on a truncated conformal expansion, which is not too surprising in view of the fact that it is only moments (or Gegenbauer moments) that are

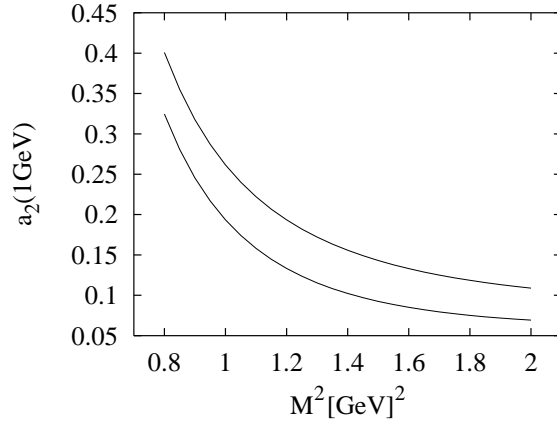


Figure 7:  $a_2(1\text{GeV})$  as function of the Borel parameter  $M^2$ . The spread between the two curves corresponds to the uncertainty induced by varying the input parameters within their error margins.  $s_0$  is fixed at  $1.8 \text{ GeV}^2$ .

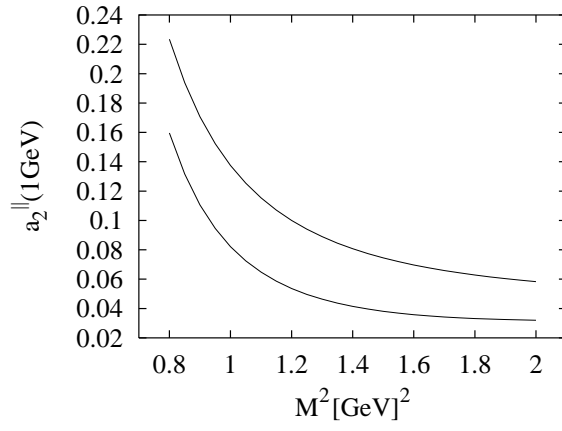


Figure 8:  $a_2^{\parallel}(1\text{GeV})$ ; notations and conventions as above.

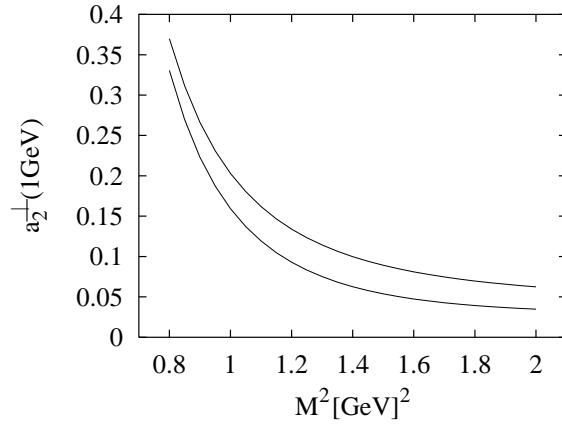


Figure 9:  $a_2^{\perp}(1\text{GeV})$ ;  $s_0$  is fixed at  $1.2 \text{ GeV}^2$ ; other notations and conventions as above.

	$K$	$K_{\parallel}^*$	$K_{\perp}^*$
$a_1(1 \text{ GeV})$	$-0.18 \pm 0.09$	$-0.40 \pm 0.20$	$-0.34 \pm 0.18$
$a_1(2 \text{ GeV})$	$-0.15 \pm 0.07$	$-0.32 \pm 0.16$	$-0.28 \pm 0.14$
$a_1(4.8 \text{ GeV})$	$-0.13 \pm 0.06$	$-0.28 \pm 0.14$	$-0.23 \pm 0.11$
$a_2(1 \text{ GeV})$	$0.16 \pm 0.10$	$0.09 \pm 0.05$	$0.13 \pm 0.08$
$a_2(2 \text{ GeV})$	$0.11 \pm 0.07$	$0.06 \pm 0.04$	$0.10 \pm 0.06$
$a_2(4.8 \text{ GeV})$	$0.09 \pm 0.06$	$0.05 \pm 0.03$	$0.08 \pm 0.05$

Table 2: First and second Gegenbauer moments  $a_1$  and  $a_2$  of the leading-twist DAs of  $K$  and  $K^*$  mesons evaluated at different scales.

accessible to direct calculation to date. Truncation after the first few moments implies that only gross features of the DAs are taken into account, which is sufficient if physical amplitudes are obtained by convoluting with a smooth perturbative scattering amplitude – or if the characteristic scale of the process is large enough for the logarithmic damping of high-order Gegenbauer moments to have become effective. In this paper we assume that this is indeed the case and that a truncation after  $n = 2$  yields a meaningful approximation. We then obtain the following model for the three leading-twist DAs of strange mesons containing an s quark:

$$\phi_K(u, \mu) = 6u(1-u) \left[ 1 + 3a_1(\mu)(2u-1) + a_2(\mu) \frac{3}{2} (5(2u-1)^2 - 1) \right]. \quad (3.10)$$

To obtain the DAs for mesons with an s antiquark, one has to replace  $u \leftrightarrow 1-u$ . The values of the Gegenbauer moments  $a_1$  and  $a_2$  are calculated from QCD sum rules at the low scale 1 GeV and scaled up to 2 and 4.8 GeV, to LO accuracy;<sup>1</sup> numerical values are collected in Tab. 2. Eq. (3.10) is ready for use in phenomenological applications.

## 4 Summary and Conclusions

We have presented a comprehensive study of SU(3) breaking corrections to the decay constants and the leading-twist light-cone distribution amplitudes of  $K$  and  $K^*$  mesons from QCD sum rules. We have corrected a few computational mistakes in previous works and included radiative corrections to the leading contributions to odd moments of the distribution amplitudes. Our main results are summarised in Eqs. (3.7), (3.10) and Tab. 2. We find in particular that the first Gegenbauer moments are *negative* for all three DAs, in contrast to previous determinations. The decay constants  $f_K$  and  $f_K^{\parallel}$  can be extracted from experiment with good accuracy, whereas  $f_K^{\perp}$  has to be determined theoretically. Its accuracy is limited mainly by the uncertainty of the input parameters, whereas the accuracy of the Gegenbauer moment  $a_1$  is in addition limited by the lack of a proper plateau region in the Borel parameter, which suggests that the uncertainty is dominated

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<sup>1</sup>NLO scaling mixes all Gegenbauer moments, so that to this accuracy the model (3.10) receives radiatively generated corrections in  $a_{n>2}$ .



by systematics. The latter limitation also affects  $a_2$ . This implies that further refinement of the sum rules by calculating even higher order corrections is not likely to improve their accuracy. Any decisive improvement has to come from a different method, for which lattice QCD appears to be a natural candidate. It is to be hoped that the preliminary results for the second moment of the twist-2 pion DA reported in [4] will be improved and also extended to the  $K$  and other mesons.

Eq. (3.10) and Tab. 2 present an approximation to the full DA to NLO in the conformal expansion. A question we have not touched upon in this paper is the suitability of this approximation for calculating physical amplitudes of type

$$A_{\text{phys}} \sim \int_0^1 du \phi(u) T(u),$$

where  $T(u)$  is a perturbative scattering amplitude. Obviously the answer to this question depends on the functional dependence of  $T$  on  $u$ . The standard argument to justify the validity of a truncated conformal expansion is that for sufficiently smooth functions  $T(u)$  higher order Gegenbauer polynomials in  $\phi(u)$  with their highly oscillatory behaviour are effectively washed out. It is also argued that, for large scales, higher order Gegenbauer moments are suppressed by renormalisation group scaling, so that for most applications the asymptotic DA is sufficient. These arguments are usually presented in a qualitative rather than a quantitative way and it would be interesting to study, within a well-defined model that does not rely on conformal expansion, how sensitive physical amplitudes actually are to a truncation of the conformal expansion, and at what scales logarithmic damping becomes effective. These questions will be addressed in a future publication.

## Acknowledgements

We are grateful to V. Sanz for collaboration in the early stages of this work.

## Appendix

### A Results for Nondiagonal Correlation Functions

We calculate the correlation function

$$\Pi_M = i \int d^4y e^{iqy} \langle 0 | T \bar{q}(y) \Gamma_2 s(y) \bar{s}(0) \Gamma_1 [0, z] q(z) | 0 \rangle, \quad (\text{A.1})$$

where  $\Gamma_{1,2}$  are Lorentz-structures suitable for the projection onto the meson  $M$ ,  $M \in \{K, K_\perp^*, K_\parallel^*\}$ ,  $z^2 = 0$  is a light-like vector and

$$[0, z] = P \exp \left\{ -ig \int_0^1 dt z_\mu A^\mu(tz) \right\}$$

the path-ordered gauge-link joining the quark fields in the bilinear  $\bar{s}(0) \Gamma_1 q(z)$ . The above sign-convention for  $g$  implies that the covariant derivative is given by  $D_\mu = \partial_\mu - ig A_\mu$ .

	$\Gamma_1$	$\Gamma_2$	projection $\mathcal{P} : \Pi_M \rightarrow \mathcal{P}\Pi_M$
$K$	$\hat{z}\gamma_5$	$\gamma_5$	$-1/(q \cdot z)$
$K_{\parallel}^*$	$\hat{z}$	$\sigma_{\mu\nu} q^\mu z^\nu$	$i/(q \cdot z)^2$
$K_{\perp}^*$	$\sigma_{\mu\nu}$	$\gamma_\alpha$	$\frac{1}{D-2} \frac{-i}{(q \cdot z)^2} \{(q \cdot z) g^{\mu\alpha} z^\nu - q^\mu z^\nu z^\alpha\}$

Table A: Dirac structures used in Eq. (A.1). The choice of  $\Gamma_{1,2}$  together with the projection  $\mathcal{P}$  projects onto the twist-2 structure and eliminates factors  $i(q \cdot z)$  picked up by the traces.  $D$  is the number of dimensions in dimensional regularisation. We use the notation  $\hat{a} = a_\mu \gamma^\mu$  for an arbitrary 4-vector  $a$ .

Odd Gegenbauer moments are most conveniently determined from nondiagonal correlation functions, involving one chiral-odd and one chiral-even current. To be specific, we choose the currents given in Tab. A and perform an operator product expansion of  $\Pi_M$  up to dimension 5 condensates. We express the perturbative results in the form

$$\mathcal{P}\Pi_M^{\text{pert.th}} = -\frac{3}{4\pi^2} \bar{m}_s(\mu) \ln \frac{-q^2}{\mu^2} \int_0^1 e^{i\bar{u}(q \cdot z)} \bar{u} \pi_M(u; \mu) + \text{terms analytic in } q^2,$$

where  $u$  is the momentum fraction carried by the s quark in the  $K$  or  $K^*$  meson;  $\bar{u} = 1 - u$  is the momentum fraction carried by the u or d antiquark. Note that one has to define  $u$  in such a way that the exponential reads  $\exp(i\bar{u}(q \cdot z))$ , rather than  $\exp(iu(q \cdot z))$ , as it has to match the corresponding factor in the hadronic parametrisation of the correlation function, cf. Eq. (3.5). The  $O(\alpha_s)$  corrections are new, and we find the following expressions:

$$\begin{aligned} \pi_K &= 1 + C_F \frac{\alpha_s}{4\pi} \left\{ \frac{51 - 2\pi^2}{3} + \ln \frac{-q^2}{\mu^2} \left( -\frac{9}{2} - \ln u \right) + \frac{2(-3 + 4u) \ln u}{\bar{u}} - \frac{\ln^2 u}{\bar{u}} \right. \\ &\quad \left. + (-3 - 2 \ln u) \ln \bar{u} + 2 \ln^2 \bar{u} + 4 L_2(u) + 2 L_2\left(-\frac{u}{\bar{u}}\right) - 2 \frac{u}{\bar{u}} \left( L_2(\bar{u}) - L_2\left(-\frac{\bar{u}}{u}\right) \right) \right\}, \\ \pi_{K_{\parallel}^*} &= 1 + C_F \frac{\alpha_s}{4\pi} \left\{ \frac{33 - 2\pi^2}{3} + 2 \ln^2 \bar{u} - (3 + 2 \ln u) \ln \bar{u} - \left( \frac{5}{2} + \ln u \right) \ln \frac{-q^2}{\mu^2} \right. \\ &\quad \left. - \frac{2(3 - 4u)}{\bar{u}} \ln u - \frac{\ln^2 u}{\bar{u}} - \frac{2u L_2(\bar{u})}{\bar{u}} - \frac{2u L_2(-\bar{u}/u)}{\bar{u}} + 4 L_2(u) + 2 L_2(-u/\bar{u}) \right\}, \\ \pi_{K_{\perp}^*} &= 1 + C_F \frac{\alpha_s}{4\pi} \left\{ \frac{33 - 2\pi^2}{3} + 2 \ln^2 \bar{u} - 3 \frac{2 - u}{\bar{u}} \ln u - \frac{1 + u}{\bar{u}} \ln^2 u - \left( 3 + \frac{\ln u}{\bar{u}} \right) \ln \frac{-q^2}{\mu^2} \right. \\ &\quad \left. - \left( 3 + \frac{2 \ln u}{\bar{u}} \right) \ln \bar{u} - \frac{2u}{\bar{u}} L_2(\bar{u}) + 4 L_2(u) + 2 L_2(-u/\bar{u}) \right\}. \end{aligned} \quad (\text{A.2})$$

	$\Gamma_1$	$\Gamma_2$	projection $\mathcal{P} : \Pi_M \rightarrow \mathcal{P}\Pi_M$
$K$	$\hat{z}\gamma_5$	$\hat{z}\gamma_5$	$1/(q \cdot z)^2$
$K_{\parallel}^*$	$\hat{z}$	$\hat{z}$	$1/(q \cdot z)^2$
$K_{\perp}^*$	$\sigma_{\mu\nu} z^\nu$	$\sigma^{\mu\rho} z_\rho$	$\frac{1}{2-D} \frac{1}{(q \cdot z)^2}$

Table B: Dirac structures and projections used for calculating even moments from diagonal correlation functions

A collection of loop integrals necessary to perform these calculations can be found in App. C. We have checked that the above expressions reproduce the correct anomalous dimensions for the Gegenbauer moments,  $\gamma_{(n)}^{\parallel}$  and  $\gamma_{(n)}^{\perp}$ , Eq. (2.6). We have also checked that in the local limit, i.e.  $z \rightarrow 0$ ,  $\exp(i\bar{u}(q \cdot z)) \rightarrow 1$ , the expressions for  $K_{\parallel}^*$  and  $K_{\perp}^*$  agree:  $\mathcal{P}\Pi_{K_{\parallel}^*} \equiv \mathcal{P}\Pi_{K_{\perp}^*}$ . Note that the sign of the leading term differs with respect to [8, 5].

We have also calculated  $O(\alpha_s)$  corrections to the condensate terms and find

$$\begin{aligned}
\mathcal{P}\Pi_M^{\langle \bar{q}q \rangle} = & \frac{1}{q^2} \int_0^1 du e^{i\bar{u}(q \cdot z)} \left[ \langle \bar{s}s \rangle \left( \delta(u) + \frac{\alpha_s}{4\pi} C_F \left\{ \delta(u) \left( 5 \ln \frac{-q^2}{\mu^2} - 3 \right. \right. \right. \right. \\
& + (1 - \delta_{M,\parallel}) \left( 2 - 3\delta_{M,\perp} + (3\delta_{M,\perp} - 4) \ln \frac{-q^2}{\mu^2} \right) \Big) - 2\bar{u} (1 - \delta_{M,\perp}) \\
& \times \left( 1 + 2\delta_{M,\parallel} + \ln(u\bar{u}) + \ln \frac{-q^2}{\mu^2} \right) + 2 \left[ \frac{\bar{u}}{u} \left( 2 - \ln(u\bar{u}) - \ln \frac{-q^2}{\mu^2} \right) \right]_+ \Big\} \Big) \\
& + \langle \bar{q}q \rangle (u \leftrightarrow \bar{u}) \Big], \tag{A.3}
\end{aligned}$$

where the  $[ ]_+$  prescription is defined as

$$[f(u)]_+ = f(u) - \delta(u_0) \int_0^1 dv f(v),$$

if  $f$  has a simple pole (modulo logarithms) at  $0 \leq u_0 \leq 1$ .

For the contribution from the dimension 5 mixed condensates  $\langle \bar{q}\sigma g G q \rangle$  and  $\langle \bar{s}\sigma g G s \rangle$ , we restrict ourselves to the tree-level contribution and find

$$\mathcal{P}\Pi_M^{\langle \bar{q}\sigma g G q \rangle} = \frac{1}{3q^4} \int_0^1 du e^{i\bar{u}(q \cdot z)} \left[ \langle \bar{s}\sigma g G s \rangle \{ (\delta_{M,\parallel} + \delta_{M,\perp}) \delta(u) + \delta'(u) \} + \langle \bar{q}\sigma g G q \rangle \{ u \leftrightarrow \bar{u} \} \right]. \tag{A.4}$$

## B Sum Rules for Moments

The Dirac structures and projections used for calculating the diagonal correlation function Eq. (3.4) are collected in Tab. B.

The nonperturbative corrections to the sum rule for even moments of the  $K$  can be extracted from [5]. We have recalculated the SU(3) breaking terms explicitly and find agreement with Ref. [5]. The perturbative terms can be extracted from Ref. [19]. We have also recalculated these terms in the nonlocal operator formalism and confirm the result quoted in [19]. The complete sum rule for even Gegenbauer moments of the  $K$  reads

$$\begin{aligned}
& \frac{3(n+1)(n+2)}{2(2n+3)} f_K^2 a_n(\mu) e^{-m_K^2/M^2} = \frac{1}{2\pi^2} \frac{\alpha_s}{\pi} M^2 \left(1 - e^{-s_0^{\parallel}/M^2}\right) \\
& \times \int_0^1 du u \bar{u} C_n^{3/2}(2u-1) \ln^2 \frac{u}{\bar{u}} + \frac{m_s \langle \bar{s}s \rangle}{2M^2} (n+1)(n+2) \\
& + \frac{1}{24M^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle (n+1)(n+2) - \frac{m_s \langle \bar{s}\sigma gGs \rangle}{24M^4} n(n+1)(n+2)(n+3) \\
& + \frac{8\pi\alpha_s}{9} \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle}{M^4} (n+1)(n+2) + \frac{4\pi\alpha_s}{81} \frac{\langle \bar{q}q \rangle^2 + \langle \bar{s}s \rangle^2}{M^4} (n+1)^2(n+2)^2, \quad (\text{B.1})
\end{aligned}$$

We have also checked the nonperturbative terms in [9] and could not confirm the terms in  $m_s \langle \bar{s}\sigma gGs \rangle$ . We thus find it appropriate to present here the (hopefully) correct sum rules:

$$\begin{aligned}
& \frac{3(n+1)(n+2)}{2(2n+3)} (f_K^{\parallel})^2 a_n^{\parallel}(\mu) e^{-m_{K^*}^2/M^2} = \\
& = \frac{1}{2\pi^2} \frac{\alpha_s}{\pi} M^2 \left(1 - e^{-s_0^{\parallel}/M^2}\right) \int_0^1 du u \bar{u} C_n^{3/2}(2u-1) \ln^2 \frac{u}{\bar{u}} \\
& + \frac{1}{2M^2} m_s \langle \bar{s}s \rangle (n+1)(n+2) + \frac{1}{24M^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle (n+1)(n+2) \\
& - \frac{1}{24M^4} m_s \langle \bar{s}\sigma gGs \rangle n(n+1)(n+2)(n+3) - \frac{8\pi\alpha_s(\mu)}{9M^4} \langle \bar{q}q \rangle \langle \bar{s}s \rangle (n+1)(n+2) \\
& + \frac{4\pi\alpha_s}{81M^4} (\langle \bar{q}q \rangle^2 + \langle \bar{s}s \rangle^2) (n+1)^2(n+2)^2, \quad (\text{B.2})
\end{aligned}$$

$$\begin{aligned}
& \frac{3(n+1)(n+2)}{2(2n+3)} (f_{K^*}^T(\mu))^2 a_n^{\perp}(\mu) e^{-m_{K^*}^2/M^2} = \\
& = \frac{1}{2\pi^2} \frac{\alpha_s}{\pi} M^2 \left(1 - e^{-s_0^{\perp}/M^2}\right) \int_0^1 du u \bar{u} C_n^{3/2}(2u-1) \left( \ln u + \ln \bar{u} + \ln^2 \frac{u}{\bar{u}} \right) \\
& + \frac{1}{24M^2} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle (n^2 + 3n - 2) - \frac{1}{24M^4} m_s \langle \bar{s}\sigma gGs \rangle (n+1)(n+2)(n^2 + 3n + 4) \\
& + \frac{4\pi\alpha_s}{81M^4} (\langle \bar{q}q \rangle^2 + \langle \bar{s}s \rangle^2) (n-1)(n+1)(n+2)(n+4) + \frac{1}{2M^2} m_s \langle \bar{s}s \rangle (n+1)(n+2). \quad (\text{B.3})
\end{aligned}$$

Sum rules for odd moments are obtained from the formulas given in App. A by Borel transforming and replacing  $\exp(i\bar{u}(q \cdot z)) \rightarrow C_n^{3/2}(2u - 1)$ . In this way, and using the properties of Gegenbauer polynomials as for instance collected in [20], which amount to the following replacements:

$$\begin{aligned}\delta(u) &\rightarrow (-1)^n \frac{1}{2} (n+1)(n+2), & \delta(\bar{u}) &\rightarrow \frac{1}{2} (n+1)(n+2), \\ \delta'(u) &\rightarrow (-1)^n \frac{1}{4} n(n+1)(n+2)(n+3), & \delta'(\bar{u}) &\rightarrow \frac{1}{4} n(n+1)(n+2)(n+3),\end{aligned}$$

we find for the  $O(\alpha_s^0)$  contributions for odd  $n$ :

$$\begin{aligned}\mathcal{B}_{\text{sub}} \mathcal{P} \Pi_K^{(n)} &= -\frac{3}{8\pi^2} m_s \left(1 - e^{-s_0/M^2}\right) + \frac{1}{2M^2} (n+1)(n+2)(\langle \bar{s}s \rangle - \langle \bar{q}q \rangle) \\ &\quad + \frac{1}{12M^4} n(n+1)(n+2)(n+3)(\langle \bar{q}\sigma g G q \rangle - \langle \bar{s}\sigma g G s \rangle), \\ \mathcal{B}_{\text{sub}} \mathcal{P} \Pi_{K_{\parallel}}^{(n)} &= -\frac{3}{8\pi^2} m_s \left(1 - e^{-s_0/M^2}\right) + \frac{1}{2M^2} (n+1)(n+2)(\langle \bar{s}s \rangle - \langle \bar{q}q \rangle) \\ &\quad + \frac{1}{12M^4} (n+1)^2(n+2)^2(\langle \bar{q}\sigma g G q \rangle - \langle \bar{s}\sigma g G s \rangle), \\ \mathcal{B}_{\text{sub}} \mathcal{P} \Pi_{K_{\perp}^*}^{(n)} &= \mathcal{B}_{\text{sub}} \mathcal{P} \Pi_{K_{\parallel}^*}^{(n)}.\end{aligned}\tag{B.4}$$

Note that the sign for the leading order perturbative contribution is different with respect to (C.9) and (C.10) in Ref. [9] and also with respect to [8, 5]. We have carefully checked that we obtain the correct sign in the local limit  $z \rightarrow 0$  and that our result stays unchanged for a different choice of the position of the nonlocal current in coordinate space, for instance  $\bar{s}(0)\Gamma_1[0, z]q(z) \rightarrow \bar{s}(-z)\Gamma_1[-z, z]q(z)$ . As it turns out, also the mixed condensate-contributions differ from the formulas given in [9].

To obtain the  $O(\alpha_s)$  corrected sum rules for odd moments from the formulas given in App. A, we also need the following continuum-subtracted Borel transforms:

$$\begin{aligned}\mathcal{B}_{\text{sub}} \ln \frac{-q^2 - i0}{\mu^2} &= -\left(1 - e^{-s_0/M^2}\right), \\ \mathcal{B}_{\text{sub}} \ln^2 \frac{-q^2 - i0}{\mu^2} &= -\frac{1}{M^2} \int_0^{s_0} ds e^{-s/M^2} 2 \ln \frac{s}{\mu^2}, \\ \mathcal{B}_{\text{sub}} \frac{1}{q^2} \ln \frac{-q^2 - i0}{\mu^2} &= \frac{1}{M^2} \left\{ \gamma_E - \ln \frac{M^2}{\mu^2} - \text{Ei} \left( -\frac{s_0}{M^2} \right) \right\},\end{aligned}$$

which complete the set of formulas needed to translate the correlation functions obtained in the previous appendix into QCD sum rules for odd moments.

$$\begin{array}{c} \text{Diagram 1: } \text{Line } q \text{ enters a vertex } \Gamma_1 \text{ from the left. Two lines emerge: } k \text{ (up-right) and } k-q \text{ (down-right).} \\ = \Gamma_1 e^{-i(k-q)z} \\ \text{Diagram 2: } \text{Line } q \text{ enters a vertex } \Gamma_2 \text{ from the left. A wavy line labeled } l, \mu \text{ connects } \Gamma_2 \text{ to another vertex.} \\ = -igz_\mu \Gamma_2 \int_0^1 dt e^{-ikz - itlz} \end{array}$$

Figure 10:  $O(g_s^0)$  and  $O(g_s^1)$  Feynman rules induced by the nonlocal current in (A.1).

## C Loop Integrals

To the benefit of apprentices, and also for future reference, we collect in this appendix relevant one- and two-loop integrals. The main difference as compared to usual calculations is the nonlocal vertex induced by  $\bar{s}(0)\Gamma_1[0, z]q(z)$ , which, to order  $\alpha_s$ , gives rise to the Feynman rules shown in Fig. 10.

The master one-loop integral is given by ( $z^2 = 0$ ,  $D = 4 + 2\epsilon$ ):

$$\begin{aligned}
\int [d^L k] e^{-i(k-q)z} \frac{(kz)^a}{(k^2)^\alpha ((k-q)^2)^\beta} &= (-1)^{\alpha+\beta} (-q^2)^{2-\alpha-\beta+\epsilon} (q \cdot z)^a \frac{\Gamma(\alpha + \beta - 2 - \epsilon)}{\Gamma(\alpha)\Gamma(\beta)} \\
&\times \int_0^1 e^{iu(q \cdot z)} u^{1-\beta+\epsilon} \bar{u}^{1-\alpha+a+\epsilon}, \quad (C.1)
\end{aligned}$$

where  $[d^L k] i/(4\pi)^{D/2} \equiv d^D k/(2\pi)^D$ .

As for two-loop integrals, one needs for instance

$$\begin{aligned}
\int_0^1 dv \int [d^L k] [d^L l] e^{-i(k-qz)} e^{-iv(l-k)z} \frac{(kz)^a (lz)^b}{k^2 l^2 (k-l)^2 (k-q)^2} &= \frac{\Gamma(-\epsilon)\Gamma(-2\epsilon)}{\Gamma(1-\epsilon)} (q \cdot z)^{a+b} (-q^2)^{2\epsilon} \\
&\times \frac{1}{i(q \cdot z)} \int_0^1 du e^{iu(q \cdot z)} \left\{ u^{2\epsilon} \bar{u}^{b+\epsilon} \int_0^1 dy y^\epsilon \bar{y}^{-1+\epsilon} ((1-uy)^{a-1} - \bar{u}^{a-1}) \right. \\
&\left. + \frac{\Gamma(\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)} u^{2\epsilon} \bar{u}^{a+b+\epsilon-1} - \frac{\Gamma(\epsilon)\Gamma(b+1+\epsilon)}{\Gamma(b+1+2\epsilon)} u^\epsilon \bar{u}^{a+b-1+2\epsilon} \right\}. \quad (C.2)
\end{aligned}$$

Generally, all loop-integrals with additional exponentials can be calculated conveniently using Feynman parameters. The calculation is further simplified by the fact that one only needs the imaginary part in  $q^2$ , which implies that finite integrals need not be calculated. “Overlapping exponentials” like for instance

$$\int_0^1 dx \int_0^1 dy \exp(i(x + y\bar{x})(q \cdot z)) f(x, y)$$

can be rewritten as

$$\int_0^1 du \exp(i\bar{u}(q \cdot z)) \int_0^1 dx \int_0^1 dy \delta(x + y\bar{x} - \bar{u}) f(x, y),$$

which allows one to reduce all contributions to the canonical form.

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